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THEORY OF THE JET SYPHON

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SUMMARY

A new approach to the theory of the mixing of two currents in an injector is presented which deals with an incompressible ideal fluid.

The equations of continuity and of motion of a single flow (assumed one dimensional), although well known, are demonstrated in order to facilitate the understanding of the modified theory.

Without advancing any new hypotheses as to the physical nature of the mixing phenomena, the theory shows new potentialities in an appropriate shaping of the form of the walls of the mixing zone so as to improve the jet-syphon efficiency beyond that heretofore theoretically predicted. This improvement is shown in spite of the fact that no mathematical optimum conditions theoretically exist.

A few examples of ways to improve jet-syphon efficiency are indicated. Application of the new theory to the case of a constant-pressure ejector thrust augmenter is presented in an appendix.

INTRODUCTION

The theory set forth in this report is an attempt to give a new viewpoint on the problem of the mixing of two currents in an injector (i.e., jet syphon), as compared with the theories now existing and accepted. This new approach is intended to clear up some ambiguity in the presentation of the problem in certain textbooks and papers on hydraulics in which the interpretation of the law of conservation of momentum seems to be oversimplified, in that the external forces (more particularly, reaction of the channel wall) are disregarded in constructing the mathematical expression for the balance of forces and quantities of motion in the general case of flow with variable pressure along the mixing zone.

The actual phenomena occurring in the channel during the mixing are very complicated and cannot be explicitly elucidated without the introduction of certain physical hypotheses concerning the behavior of the microscopic particles of the liquid. It is not intended to give in this paper such a detailed physical explanation of these actual phenomena. A few general deductions as to the behavior of the current, its velocity, pressure, and so forth, may be drawn, however, from the general principles of conservation (of mass, of energy, and of momentum), if some additional abstractive assumptions are made. This is done in the section "Introduction to New Theory of Jet Syphon." An outline of certain principles of hydrodynamics, namely, the equation of continuity and the equation of motion, is presented first to facilitate the understanding of the modified theory.

This work was done at the Ecole Polytechnique (University of Montreal) and has been made available to the National Advisory Committee for Aeronautics for publication because of its general interest.

SYMBOLS

A	cross-sectional area of channel, sq ft
a	acceleration, ft/sec^2
c	velocity, ft/sec
E_{tot}	total (kinetic plus pressure) energy of fluid, ft-lb
ΔE_{tot}	total energy loss due to shock
F	external mass force, lb
g	terrestrial acceleration, ft/sec^2
H	pump total head, ft
n	weight output ratio; $n_1 = \dfrac{w_1}{w_1 + w_2}$ for primary stream and $n_2 = 1 - n_1$ for secondary stream
p	static pressure, lb/sq ft
r	radius, ft
s,x,l	channel length, ft
ds	infinitesimal element of channel length

Th	thrust, lb
$(Th)_o$	thrust of single (primary) stream, lb
w	weight output, lb/sec; w_1 for primary and w_2 for secondary stream
X	potential of mass forces, ft^2/sec^2
z	elevation in gravitational field, ft
γ	specific weight, lb/cu ft
η_{ex}	external propulsive efficiency
$(\eta_{ex})_o$	external propulsive efficiency of a single stream
η_i	injector efficiency
ρ	density, γ/g
τ	time, sec
ω	angular velocity, sec^{-1}

Dimensionless coefficients:

$$\delta = \sqrt{\frac{w_1 \sigma_{1_o}'}{w_{1_o} \sigma_1'}}$$

$$\epsilon = \frac{\eta_{ex}}{(\eta_{ex})_o}$$

$$\zeta = \frac{Th}{(Th)_o}$$

$$\lambda = \sqrt{\frac{H\gamma_o}{p_o}}$$

$$\pi = p_1/p_o$$

$\sigma = c_1''/c_1'$

$$\sigma_1' = \frac{c_1'\sqrt{\gamma_o}}{\sqrt{2gp_o}};\ \sigma_o = \frac{c_o\sqrt{\gamma_o}}{\sqrt{2gp_o}}\ \text{(in appendix only)}$$

Subscripts:

o	atmospheric conditions
1	initial cross section of mixing zone
2	final cross section of mixing zone
3	diffuser exit

Superscripts:

'	primary stream
"	secondary stream

EQUATION OF CONTINUITY OF A SINGLE FLOW

The following theory is based on the assumption that the flow is one dimensional, that is, that there is only one geometrical dimension, the length of the flow path. This unique geometrical coordinate is, however, considered as curvilinear; that is, the "axis" of the flow may be shaped arbitrarily in the three-dimensional space, provided the flow vein is sufficiently fine to consider the pressure and the velocity uniform in the whole arbitrarily chosen cross section of the vein. Such an approach is, of course, not entirely exact, but it has given fully satisfactory results in dealing with the thermodynamic problems of stationary nozzles and diffusers.

The other assumption made herein is that the fluid is ideal, that is, nonviscous and involving no friction at the channel walls.

Consider a certain flow vein whose axis is arbitrarily shaped in space (fig. 1). Let its cross section A be variable with the length s of the flow path and with time. Cutting this vein at any point whatsoever, perpendicularly to the axis, with two planes A and B which are at an infinitely small distance ds from each other and supposing that in plane A the cross section is A; the pressure, P; the velocity, c; and the density, ρ; one has in the B plane:

$$\text{Cross section:} \quad A + \frac{\partial A}{\partial s}\,ds$$

$$\text{Pressure:} \quad p + \frac{\partial p}{\partial s}\,ds$$

$$\text{Velocity:} \quad c + \frac{\partial c}{\partial s}\,ds$$

$$\text{Density:} \quad \rho + \frac{\partial \rho}{\partial s}\,ds$$

where the cross section is measured in square feet; the pressure, in pounds per square foot; and the velocity, in feet per second; and the density is

$$\rho = \gamma/g$$

where γ, in pounds per cubic foot, is the specific weight; and $g = 32.174\ \text{ft/sec}^2$ is the terrestrial acceleration.

During the infinitely small time interval $d\tau$ the amount $\rho Ac\ d\tau$ of the mass of fluid enters the vein element ds through cross section A, while another amount of mass

$$\left(\rho + \frac{\partial \rho}{\partial s}\,ds\right)\left(A + \frac{\partial A}{\partial s}\,ds\right)\left(c + \frac{\partial c}{\partial s}\,ds\right)d\tau$$

leaves the vein element through the other cross section B.

On the other hand, during the same time interval $d\tau$, the mass of the vein sector ds has increased from $\rho A\ ds$ to

$$\left(\rho + \frac{\partial \rho}{\partial \tau}\,d\tau\right)\left(A + \frac{\partial A}{\partial \tau}\,d\tau\right)ds$$

One has, therefore, arrived, under the condition of the continuity of flow, at the equality

$$\left(\rho + \frac{\partial \rho}{\partial \tau}\,d\tau\right)\left(A + \frac{\partial A}{\partial \tau}\,d\tau\right) ds - \rho A\,ds =$$

$$\rho Ac\,d\tau - \left(\rho + \frac{\partial \rho}{\partial s}\,ds\right)\left(A + \frac{\partial A}{\partial s}\,ds\right)\left(c + \frac{\partial c}{\partial s}\,ds\right) d\tau$$

as the given amount of mass cannot change.

Neglecting the infinitely small quantities of the second and third order and introducing the definition of velocity

$$c = ds/d\tau$$

there is obtained

$$\frac{\partial}{\partial \tau}(A\rho) + \frac{\partial}{\partial s}(Ac\rho) = 0 \tag{1}$$

which is the mathematical form of the principle of continuity of flow and is called the equation of continuity.

Particular cases may be easily deduced from equation (1). For instance, if the fluid is incompressible, ρ = Constant, and therefore

$$\frac{\partial A}{\partial \tau} + \frac{\partial}{\partial s}(Ac) = 0$$

Another particular case occurs when the channel cross section is constant, that is, when A = Constant, and therefore

$$\frac{\partial \sigma}{\partial \tau} + \frac{\partial}{\partial s}(c\rho) = 0$$

If, at the same time, the fluid is incompressible, one obtains $\partial c/\partial s = 0$; that is, the velocity c is only a function of time (i.e., at a given moment it is constant through the whole channel).

A further possible particular case is the steady flow, which is independent of time; that is,

$$\frac{\partial}{\partial s}(Ac\rho) = \frac{d}{ds}(Ac\rho) = 0$$

$$A c \rho = \frac{A c \gamma}{g} = \text{Constant} = \frac{A_o c_o \gamma_o}{g} \tag{2}$$

If, in addition, the fluid is incompressible, then $\gamma = \text{Constant} = \gamma_o$, and one gets

$$A c = \text{Constant} = A_o c_o \tag{2a}$$

EQUATION OF MOTION FOR A SINGLE FLOW

The principle of conservation of momentum (i.e., of quantity of motion) resulting from Newton's law which states that the force equals the product of mass and acceleration enables one to obtain the equation of motion.

The forces acting on sector ds of the vein are as follows:

(1) On the one hand, the resultant of pressure forces exerted on cross section A and on the lateral surface; this resultant force simply equals the product of the pressure p and of the projection of cross section A and the vein lateral surface on the plane of cross section B; that is,

$$p\left(A + \frac{\partial A}{\partial s} ds\right)$$

(2) On the other hand, one has, similarly,

$$\left(p + \frac{\partial p}{\partial s} ds\right)\left(A + \frac{\partial A}{\partial s} ds\right)$$

and the resultant force in the positive direction of s is

$$p\left(A + \frac{\partial A}{\partial s} ds\right) - \left(p + \frac{\partial p}{\partial s} ds\right)\left(A + \frac{\partial A}{\partial s} ds\right) \approx -A \frac{\partial p}{\partial s} ds$$

Apart from that, the external force F, called mass force, may also act on the sector ds. This force is bound with the mass of fluid and it always possesses a certain potential X such that the component of mass force in any direction n, reckoned per the unit of mass, equals the partial derivative of X in this direction, that is, $\partial X/\partial n$. One, therefore, has in the direction of motion s:

$$dF_s = \frac{\partial X}{\partial s} \rho A \, ds$$

and the resultant force will be

$$dF_s - A \frac{\partial p}{\partial s} ds = \frac{\partial X}{\partial s} \rho A \, ds - A \frac{\partial p}{\partial s} ds$$

The acceleration of the element ds in the direction of motion is

$$a = \frac{dc}{d\tau} = \frac{\partial c}{\partial s} \frac{ds}{d\tau} + \frac{\partial c}{\partial \tau} = c \frac{\partial c}{\partial s} + \frac{\partial c}{\partial \tau}$$

and the mass of the element is $\rho A \, ds$; therefore, the equation of motion (also called Bernoulli's equation) will be

$$\rho A \, ds \frac{\partial X}{\partial s} - A \frac{\partial p}{\partial s} ds = \rho A \, ds \left(c \frac{\partial c}{\partial s} + \frac{\partial c}{\partial \tau} \right)$$

or

$$\frac{\partial X}{\partial s} - \frac{1}{\rho} \frac{\partial p}{\partial s} = c \frac{\partial c}{\partial s} + \frac{\partial c}{\partial \tau} \tag{3}$$

For steady (i.e., stationary) flow there is obtained

$$\frac{\partial X}{\partial s} ds - \frac{g}{\gamma} dp = c \, dc$$

and, after integration,

$$X - g \int \frac{dp}{\gamma} = \frac{c^2}{2} + \text{Constant} \tag{4}$$

Among the mass forces, the following three may, above all others, actually occur:

(a) Gravitational force

(b) Centrifugal force

(c) Shock force

All these are "external" forces, in contrast with possible "internal" mass forces (e.g., viscosity).

The gravitational force is caused by the gravitational field of the earth and it actually occurs in heavy, incompressible liquids (e.g., water). Its presence is independent of the character of motion.

The centrifugal force may occur only in rotational motion or in a combination of rotational and progressive motions. In purely progressive motion centrifugal force does not exist, as here the radial acceleration (of the Coriolis type) is nil.

The appearance of shock forces in any actual hydraulic or aerodynamic device is, in general, unavoidable. When the fluid (either gas or liquid) stream strikes a rigid wall, either fixed or in motion, the shock force appears in the form of a reaction of the wall, and this is, of course, an external force, relative to the stream. The same reaction may occur if the liquid stream strikes another liquid stream, which will produce a reaction force external to the former liquid stream.

As is seen, the general equation of motion (eq. (3)) is independent of both cross section A and its changes in space and time, provided, as has already been pointed out, the flow vein is sufficiently thin.

Equation (4) is integrable if the law of variation of density with pressure is known and given. Insofar as gases are concerned, these variations depend on pressure and temperature, and therefore the character of thermodynamic evolution (i.e., process) must be determined. In the case of an incompressible fluid, $\gamma = \text{Constant} = \gamma_o$, equation (4) is immediately integrable, giving

$$X - \frac{g}{\gamma_o} p - \frac{c^2}{2} = \text{Constant} \tag{5}$$

where the value of the constant is to be defined from the initial, final, or boundary conditions.

The following examples of mass forces may be cited:

(a) Gravitational; it acts in the direction of the z-axis (fig. 2) and is simply

$$dF = -(\rho A\ ds)g$$

where the negative sign means that the force is directed down. It follows that $X = -gz$; thus, for an incompressible fluid,

$$z + \frac{p}{\gamma_o} + \frac{c^2}{2g} = \text{Constant} = z_o + \frac{p_o}{\gamma_o} + \frac{{c_o}^2}{2g}$$

(b) Centrifugal; it acts in the direction of the radius r and is proportional to the centrifugal acceleration

$$a = \omega^2 r$$

where $\omega = 2\pi n/60$ is the angular velocity, n designating the number of revolutions per minute. Thus the mass force becomes

$$dF = (\rho A\ ds)a$$

that is,

$$X = \frac{\omega^2 r^2}{2} = \frac{\pi^2 n^2}{1,800} r^2$$

and therefore

$$\frac{\pi^2 n^2}{1,800} r^2 - \frac{p}{\gamma_o} - \frac{c^2}{2g} = \text{Constant}$$

This equation is known and applied to rotating, dynamically acting machines (turbines, pumps, etc.).

(c) As for the shock force, the case of the exchange of shocks between the two flowing liquids, as occurs in a jet syphon, is the main object of study of this paper, and it is discussed in detail in the following section.

INTRODUCTION TO NEW THEORY OF JET SYPHON

The mixing of the two currents in an injector (i.e., jet syphon) is a typical example of hydrodynamic shock provided the velocities of the two currents are initially different, that is, provided they are different at the first moment of the meeting of the two currents. The following computation is restricted to the case of an incompressible fluid in both currents.

Let the exit of the internal channel I be placed at $x = 0$ (fig. 3), where its cross-sectional area is A_1' and the velocity is c_1'. At the same cross section the external channel E has a cross-sectional area A_1'' and the velocity of the flowing fluid is $c_1'' \neq c_1'$. Both fluids are considered to be incompressible and of the same specific weight γ_o.

As was stated in the "Introduction," a few abstractive, but logically admissible, assumptions must be made in order to make it possible to draw conclusions from the three general principles of conservation. These assumptions are as follows:

(1) The two currents are considered to atomize, one into the other, and to intermingle so perfectly that the exchange of shocks is not only between the surfaces but also between the two whole masses.

(2) Obviously, the "external" shock force F'' acting on the secondary fluid, being merely the action of the primary fluid, is always in equilibrium with the "external" shock force F' acting on the primary fluid, which is merely the action of the secondary fluid; thus, $F'' = -F'$.

(3) In any given cross section, for example, A, at distance x, the pressure throughout the cross section, that is, the pressure p' of the primary fluid, equals the pressure p'' of the secondary fluid, as it would be senseless to assume two different pressures in two fluids which are perfectly mingled one with the other. However, the corresponding velocities c' and c'' are different, until all possible shocks have been exchanged. This occurs for the first time at cross section A_2 at distance l, where c' and c'' become equal (e.g., c_2).

Of course, $A = A' + A''$ and, according to the law of continuity, the weight outputs will be w_1 and w_2, defined as follows:

$$w_1 = \gamma_o A_1' c_1' = \gamma_o A' c'$$

$$w_2 = \gamma_o A_1'' c_1'' = \gamma_o A'' c''$$

According to the section "Equation of Motion for a Single Flow," the mass force is

$$dF = \frac{\gamma_o}{g} A \frac{\partial X}{\partial x} dx$$

that is,

$$\frac{1}{\gamma_o} \int \frac{dF}{A} - \frac{p}{\gamma_o} - \frac{c^2}{2g} = \text{Constant}$$

provided that, in general, the cross-sectional area A is variable with the channel length x.

Applying this last equation consecutively to the two currents, one gets

$$\frac{1}{\gamma_o}\int\frac{dF'}{A'} - \frac{p'}{\gamma_o} - \frac{(c')^2}{2g} = \text{Constant}$$

$$\frac{1}{\gamma_o}\int\frac{dF''}{A''} - \frac{p''}{\gamma_o} - \frac{(c'')^2}{2g} = \text{Constant}$$

Differentiating and introducing $A' = \frac{w_1}{\gamma_o c'}$ and $A'' = \frac{w_2}{\gamma_o c''}$, as well as the condition $p'' = p' = p$, give

$$F' = \frac{w_1}{\gamma_o}\frac{dp}{c'} + \frac{w_1}{g}\,dc'$$

$$F'' = \frac{w_2}{\gamma_o}\frac{dp}{c''} + \frac{w_2}{g}\,dc''$$

As, however, $F'' = -F'$, this reduces to

$$\frac{1}{\gamma_o}\left(\frac{w_1}{c'} + \frac{w_2}{c''}\right) dp + \frac{w_1}{g}\,dc' + \frac{w_2}{g}\,dc'' = 0 \tag{6}$$

One of the simplest solutions of this equation is obtained in the particular case of constant pressure throughout the channel, that is, $p = \text{Constant} = p_o$. In this case

$$w_1\,dc' + w_2\,dc'' = 0$$

$$w_1c' + w_2c'' = \text{Constant} = w_1c_1' + w_2c_1'' \tag{7}$$

The final velocity is here

$$c_2'' = c_2' = c_2 = \frac{w_1c_1' + w_2c_1''}{w_1 + w_2} \tag{8}$$

Introducing the conditions of continuity, that is,

$$A' = \frac{w_1}{\gamma_o c'}$$

$$A'' = \frac{w_2}{\gamma_o c''}$$

$$A = A' + A''$$

the following expression is obtained for the channel cross-sectional area:

$$A = \frac{(w_1 c_1' + w_2 c_1'')w_1 - \left(w_1^2 - w_2^2\right)c'}{\gamma_o c'\left[\left(w_1 c_1' + w_2 c_1''\right) - w_1 c'\right]} \tag{9}$$

wherefrom it may be deduced that A diminishes with the diminishing velocity c' and attains a minimum value when c' takes its final value given by equation (8). The form of the channel may be deduced from equation (9) only if an additional assumption is made as to the variation of c' with the channel length. Of course, the simplest assumption will be that c' diminishes linearly with the increasing distance x from the duct entry, namely,

$$c' = c_1' - \frac{w_2\left(c_1' - c_1''\right)}{\left(w_1 + w_2\right)l}x$$

$$c'' = c_1'' + \frac{w_1\left(c_1' - c_1''\right)}{\left(w_1 + w_2\right)l}x$$

where l is the total length of the duct.

During the mixing of two currents of initially different speeds, a certain loss of energy occurs because of shocks which do not contribute to the exchange of momentum in the direction of flow. These shocks do certain work which finally degenerates into heat (increase of temperature), the molecular motion induced by these shocks becoming disordered. Therefore, the sum of the kinetic and potential energies of the two fluids must gradually diminish. This loss may be expressed as

$$\Delta E = \left[\frac{w_1}{2g}(c_1')^2 + \frac{w_2}{2g}(c_1'')^2 + (w_1 + w_2)\frac{p_1}{\gamma_o}\right] -$$

$$\frac{w_1}{2g}(c')^2 - \frac{w_2}{2g}(c'')^2 - (w_1 + w_2)\frac{p}{\gamma_o}$$

Thus, the total energy loss in the injector after the velocity becomes uniform and equal in the two currents is

$$\Delta E_{tot} = \frac{w_1}{2g}\left[(c')^2 - c_2^2\right] - \frac{w_2}{2g}\left[c_2^2 - (c_1'')^2\right] + (w_1 + w_2)\frac{(p_1 - p_2)}{\gamma_o} \tag{10}$$

The total energy of the fluid (kinetic energy plus pressure energy) is

$$E_{tot} = \frac{w_1}{2g}(c_1')^2 + \frac{w_2}{2g}(c_1'')^2 + (w_1 + w_2)\frac{(p_1 - p_o)}{\gamma_o} \tag{11}$$

Thus, the efficiency of the injector will be

$$\eta_i = 1 - \frac{\Delta E_{tot}}{E_{tot}} \tag{12}$$

provided the flow and the fluids are perfect, that is, provided no friction or other losses are involved.

In the particular case of constant pressure throughout the channel, as quoted above, $p = \text{Constant} = p_o$, and

$$\Delta E_{tot} = \frac{w_1 w_2(c_1' - c_1'')^2}{2g(w_1 + w_2)}$$

$$E_{tot} = \frac{w_1(c_1')^2 + w_2(c_1'')^2}{2g}$$

$$\eta_i = \frac{(w_1 c_1' + w_2 c_1'')^2}{(w_1 + w_2)\left[w_1(c_1')^2 + w_2(c_1'')^2\right]}$$

The other particular case is a tube of constant cross section; that is,

$$A' + A'' = \frac{w_1}{\gamma_o c'} + \frac{w_2}{\gamma_o c''} = A = \text{Constant} = A_o = \frac{w_1}{\gamma_o c_1'} + \frac{w_2}{\gamma_o c_1''}$$

In this case equation (6) becomes easily integrable, as one of the two unknown velocities, for example, c'', may be eliminated. One gets:

$$c'' = \frac{w_2 c_1' c_1'' c'}{\left(w_1 c_1'' + w_2 c'\right)c' - w_1 c_1' c_1''}$$

$$c_2' = c_2'' = c_2 = \frac{\left(w_1 + w_2\right)c_1' c_1''}{\left(w_1 c_1'' + w_2 c_1'\right)}$$

$$\frac{g}{\gamma_o}\left(\frac{w_1}{c_1'} + \frac{w_2}{c_1''}\right)\left(p - p_1\right) + w_1\left(c' - c_1'\right) + w_2\left(c'' - c_1''\right) = 0$$

wherefrom

$$p = p_1 + \frac{\gamma_o\left(w_1 c_1' + w_2 c_1''\right)c_1' c_1''}{g\left(w_1 c_1'' + w_2 c_1'\right)} -$$

$$\frac{\gamma_o\left[w_1\left(w_1 c_1'' + w_2 c_1'\right)c' - \left(w_1^2 - w_2^2\right)c_1' c_1''\right]c_1' c_1'' c'}{g\left(w_1 c_1'' + w_2 c_1'\right)\left[\left(w_1 c_1'' + w_2 c_1'\right)c' - w_1 c_1' c_1''\right]}$$

$$p_2 = p_1 + \frac{\gamma_o w_1 w_2 c_1' c_1''\left(c_1' - c_1''\right)^2}{g\left(w_1 c_1'' + w_2 c_1'\right)^2}$$

It may be easily demonstrated that the pressure increases steadily with decreasing c', that is, with the channel length, attaining a maximum value at the final moment, when $c' = c'' = c_2$, that is, at the end of the duct. At the same moment, assuming $p_1 = p_o$,

$$\Delta E_{tot} = \frac{w_1 w_2 \left[w_1 \left(c_1''\right)^2 + w_2 \left(c_1'\right)^2\right] \left(c_1' - c_1''\right)^2}{2g \left(w_1 c_1'' + w_2 c_1'\right)^2}$$

$$\eta_1 = \frac{\left(w_1 + w_2\right) \left[2 w_1 w_2 \left(c_1' - c_1''\right)^2 + \left(w_1^2 + w_2^2\right) c_1' c_1''\right] c_1' c_1''}{\left[w_1 \left(c_1'\right)^2 + w_2 \left(c_1''\right)^2\right] \left(w_1 c_1'' + w_2 c_1'\right)^2}$$

It may be easily proved that the efficiency is in this case generally lower than that in the case of the constant-pressure injector as studied above.

STUDY OF GENERAL CASE

Consider again the general case, that is, the case in which the duct cross section varies arbitrarily with its length. In this case the pressure will also vary along the duct and its final value, at the duct end, will be, in general, different from the initial value. Thus, to get the desirable final value of the pressure, the usual nozzle (or the usual diffuser, according to whether p_2 is greater or less than the desired value) should be applied.

It should be first remembered that the equation of motion applying to the general case has already been given as (eq. (6))

$$\frac{1}{\gamma_o}\left(\frac{w_1}{c'} + \frac{w_2}{c''}\right) dp + \frac{w_1}{g} dc' + \frac{w_2}{g} dc'' = 0$$

On the other hand, one has the following conditions of continuity of flow:

$$\left.\begin{aligned} A' &= \frac{w_1}{\gamma_o c'} \\ A'' &= \frac{w_2}{\gamma_o c''} \\ A &= A' + A'' \end{aligned}\right\} \qquad (13)$$

where the total cross-sectional area of the duct A is arbitrarily shaped along the duct length (fig. 4):

$$A = \varphi(x)$$

Putting equation (13) into equation (6) and eliminating A'', the following expression is obtained:

$$dp - \frac{w_2^{\,2}}{gA(A - A')^2}\,dA + \frac{1}{g\gamma_o}\left[\frac{w_2^{\,2}}{(A - A')^2} - \frac{w_1^{\,2}}{(A')^2}\right]\frac{dA'}{A} = 0$$

which is not integrable until an additional hypothesis is made as to changes of c' (i.e., of A') with the duct length. However, the procedure may be modified as follows: Let the decision as to possible changes of c' with the duct length be deferred until the mathematical solution of equation (6) is found (it may be next assumed that, for instance, c' diminishes linearly with the increasing duct length x), while it is assumed first that the following relation

$$A = \frac{1}{\gamma_o}\,\psi(c')$$

is arbitrarily given instead of $A = \varphi(x)$. Denoting $\frac{d\psi}{dc'} = \psi'$, it is found that

$$A'' = A - A' = \frac{1}{\gamma_o}\left(\psi - \frac{w_1}{c'}\right)$$

$$c'' = \frac{w_2 c'}{c'\psi - w_1}$$

$$dc'' = -\frac{w_2\left[(c')^2\psi' + w_1\right]}{\left(c'\psi - w_1\right)^2}\,dc'$$

Substituting into equation (6) gives

$$\frac{dp}{\gamma_o} + \frac{1}{g\psi}\left\{w_1 - \frac{w_2^{\,2}\left[(c')^2\psi' + w_1\right]}{\left(c'\psi - w_1\right)^2}\right\}dc' = 0 \qquad (14)$$

which is integrable, provided the function $\psi(c')$ is given. There is obtained

$$\frac{1}{\gamma_o}\left(p - p_1\right) + \frac{1}{g}\int_{c_1'}^{c'} \left\{w_1 - \frac{w_2^2\left[\alpha^2\psi'(\alpha) + w_1\right]}{\left[\alpha\psi(\alpha) - w_1\right]^2}\right\}\frac{d\alpha}{\psi(\alpha)} = 0 \qquad (15)$$

where α is an auxiliary variable used only for integration.

At the end of the duct, where the total equalization of velocities is first achieved, one should have:

$$c'' = c' = c_2' = c_2'' = c_2$$

$$\psi(c_2) = \gamma_o A_2 = \frac{w_1 + w_2}{c_2}$$

$$\frac{1}{\gamma_o}\left(p_2 - p_1\right) + \frac{1}{g}\int_{c_1'}^{c_2} \left\{w_1 - \frac{w_2^2\left[\alpha^2\psi'(\alpha) + w_1\right]}{\left[\alpha\psi(\alpha) - w_1\right]^2}\right\}\frac{d\alpha}{\psi(\alpha)} = 0 \qquad (15a)$$

It might be easily shown that both the particular cases as quoted in the section "Introduction to New Theory of Jet Syphon" (p = Constant and A = Constant) may be deduced from equation (14) or (15).

In designing an injector, the aim is to have the greatest possible efficiency for any given initial conditions $\left(w_1,\ w_2,\ c_1',\ \text{and}\ c_1''\right)$. The form of the function $\psi(c')$ should, therefore, be so chosen as to have the greatest possible value of

$$\eta_i = \frac{c_2^2 - 2\int_{c_1'}^{c_2}\left\{w_1 - \frac{w_2^2\left[\alpha^2\psi'(\alpha) + w_1\right]}{\left[\alpha\psi(\alpha) - w_1\right]^2}\right\}\frac{d\alpha}{\psi(\alpha)}}{\frac{w_1}{w_1 + w_2}\left(c_1'\right)^2 + \frac{w_2}{w_1 + w_2}\left(c_1''\right)^2} \qquad (16)$$

which means that in the numerator

$$J = c_2^{\,2} - 2\int_{c_1'}^{c_2} \left\{ w_1 - \frac{w_2^{\,2}\left[\alpha^2\psi'(\alpha) + w_1\right]}{\left[\alpha\psi(\alpha) - w_1\right]^2} \right\} \frac{d\alpha}{\psi(\alpha)} \tag{17}$$

the function ψ should be so chosen as to have J attaining a maximum.

In order to solve this problem correctly, the calculus of variations should be applied. The following discussion is based on the theory as given in reference 1 (ch. XXXIV, pp. 545-600).

The general problem of the calculus of variations is as follows: A form of the function $f(x)$ is looked for such that the integral

$$J = \int_{x_0}^{x_1} F(x,y,y')\,dx$$

is a maximum, where $y = f(x)$ and $y' = \frac{df}{dx}$, x_0 and x_1 being two constant limits of integration. Such a maximum does not always exist. There are several conditions for the existence of a maximum (or minimum). The first of these is that the following equation (called Euler's equation) must be fulfilled:

$$\frac{\partial^2 F}{\partial (y')^2}\,y'' + \frac{\partial^2 F}{\partial y\,\partial y'}\,y' + \frac{\partial^2 F}{\partial x\,\partial y'} - \frac{\partial F}{\partial y} = 0$$

In the present specific case the expressions

$$c_2^{\,2} = 2\int_{c_1'}^{c_2} \alpha\,d\alpha + \left(c_1'\right)^2$$

$$y = f(x) = \psi(\alpha)$$

$$F = x + \frac{w_1}{y} - \frac{w_2^{\,2}\left(x^2 y' + w_1\right)}{y\left(xy - w_1\right)^2}$$

may be substituted into Euler's equation. The final result

$$\psi(\alpha) = \frac{w_1 + w_2}{\alpha}$$

is obviously trivial because it requires that $c_1' = c_1''$.

The conclusion is that no extremum exists in the present specific case and therefore no form of the function ψ may be found giving such an extremum. There is no doubt, however, that a certain form of ψ may give a higher value of efficiency η_1 than some other form. Thus, the alternative is to try several possible mathematical forms of ψ. These forms may be chosen so as to meet the simplest mathematical operations and yet to have a certain degree of generality by introducing a few arbitrary constants, which could next be so chosen as to attain the highest value of η_1 possible in the specific case.

One of the possible forms may be, for instance

$$\psi(\alpha) = \frac{w_1c_1'' + w_2c_1'}{\left(c_1'\right)^{\mu+1}c_1''}(\alpha)^{\mu}$$

where $\mu \lesseqgtr 0$. Another is

$$\psi(\alpha) = \gamma_o A_1 + \frac{\gamma_o\left(A_2 - A_1\right)}{c_1' - c_2}\left(c_1' - \alpha\right)$$

where $A_2 = \mu A_1$ and $\mu \lesseqgtr 1$. Furthermore, it may be possible that

$$\psi(\alpha) = a - b\alpha$$

where a and b are constants.

One more possible form, which is as follows,

$$\psi(\alpha) = a + \frac{b}{\alpha} \tag{18}$$

will be studied more thoroughly below, as it seems to be relatively very simple and to give satisfactory results. It must be strongly emphasized here that there is no proof whatsoever that the form chosen is the best possible. On the contrary, it is very probable that other forms of ψ may be found, which, although probably more involved mathematically, will give still higher values of efficiency for the injector.

STUDY OF SPECIFIC FORM OF DUCT AS ASSUMED BY EQUATION (18)

Assuming the specific duct form (eq. (18))

$$\psi(\alpha) = a + \frac{b}{\alpha}$$

means that

$$A = \frac{\psi(c')}{\gamma_o} = \frac{1}{\gamma_o}\left(a + \frac{b}{c'}\right)$$

Therefore,

$$A_1 = \frac{w_1}{\gamma_o c_1'} + \frac{w_2}{\gamma_o c_1''} = \frac{1}{\gamma_o}\left(a + \frac{b}{c_1'}\right)$$

$$A_2 = \frac{w_1 + w_2}{\gamma_o c_2} = \frac{1}{\gamma_o}\left(a + \frac{b}{c_2}\right)$$

In addition, it is assumed that

$$A_2 = \mu A_1$$

where μ is an arbitrary constant whose value should be so chosen as to have the highest value of the efficiency η_i under the given circumstances. Thus:

$$c_2 = \frac{(w_1 + w_2)c_1'c_1''}{\mu\left(w_1c_1'' + w_2c_1'\right)}$$

$$a = \frac{\mu w_2\left(c_1' - c_1''\right)\left(w_1c_1'' + w_2c_1'\right)}{c_1'c_1''\left[\mu\left(w_1c_1'' + w_2c_1'\right) - \left(w_1 + w_2\right)c_1''\right]}$$

$$b = \frac{(\mu - 1)\left(w_1 + w_2\right)\left(w_1 c_1'' + w_2 c_1'\right)}{\mu\left(w_1 c_1'' + w_2 c_1'\right) - \left(w_1 + w_2\right)c_1''}$$

Substituting the above values into equation (17) and assuming, for greater simplicity in writing, that

$$\frac{w_1}{w_1 + w_2} = n_1$$

$$\frac{w_2}{w_1 + w_2} = n_2 = 1 - n_1$$

$$\frac{c_1''}{c_1'} = \sigma \quad (0 \leqq \sigma \leqq 1)$$

one gets, after integration,

$$\begin{aligned}
\frac{J}{(c_1')^2} = {} & \frac{2\sigma}{n_1^2(1 - n_1)^2(1 - \sigma)^2}\left\{n_1(1 - n_1)(1 - \sigma)\left[n_1^2 - (1 - n_1)^2\sigma\right] - n_1^3\sigma \log_e\left(n_1 + \frac{1 - n_1}{\sigma}\right) + \right. \\
& \left.(1 - n_1)^3\sigma \log_e\left(\frac{1}{n_1\sigma + 1 - n_1}\right)\right\} - \frac{2\sigma^2}{n_1^2(1 - n_1)^2(1 - \sigma)^2(n_1\sigma + 1 - n_1)}\left\{2n_1(1 - n_1)(2n_1 - 1)(1 - \sigma) - \right. \\
& \left. n_1^3\left[(n_1 + 1)\sigma + 1 - n_1\right] \log_e\left(n_1 + \frac{1 - n_1}{\sigma}\right) + (1 - n_1)^3(n_1\sigma + 2 - n_1) \log_e\left(\frac{1}{n_1\sigma + 1 - n_1}\right)\right\}\frac{1}{\mu} + \\
& \frac{\sigma^2}{n_1^2(1 - n_1)^2(1 - \sigma)^2(n_1\sigma + 1 - n_1)^2}\left\{n_1(1 - n_1)(1 - \sigma)\left[n_1(3n_1 - 1)\sigma + (1 - n_1)(3n_1 - 2)\right] - \right. \\
& \left. 2n_1^3\sigma(n_1\sigma + 1 - n_1) \log_e\left(n_1 + \frac{1 - n_1}{\sigma}\right) + 2(1 - n_1)^3(n_1\sigma + 1 - n_1) \log_e\left(\frac{1}{n_1\sigma + 1 - n_1}\right)\right\}\frac{1}{\mu^2}
\end{aligned} \tag{19}$$

$$\eta_i = \frac{J\Big/(c_1')^2}{n_1 + (1 - n_1)\sigma^2} \tag{20}$$

Now the value of μ giving the highest value of η_i should be found. Equation (20) is of the form

$$\eta_i = a_1 + \frac{a_2}{\mu} + \frac{a_3}{\mu^2}$$

where a_1, a_2, and a_3 are constants. Thus

$$\frac{d\eta_i}{d\mu} = -\frac{a_2}{\mu^2} - \frac{2a_3}{\mu^3}$$

$$\frac{d^2\eta_i}{d\mu^2} = \frac{2a_2}{\mu^3} + \frac{6a_3}{\mu^4}$$

The condition $\frac{d\eta_i}{d\mu} = 0$ will give $\mu_e = -\frac{2a_3}{a_2}$, $\left(\frac{d^2\eta_i}{d\mu^2}\right)_e = \frac{{a_2}^4}{8{a_3}^3}$, and $\left(\eta_i\right)_e = a_1 - \left({a_2}^2/4a_3\right)$. Thus a maximum of η_i will exist only if $\frac{d^2\eta_i}{d\mu^2} < 0$, that is, if $a_3 < 0$ and, consequently, $a_2 > 0$, because $\mu_e < 0$ would have no physical sense.

The correct mathematical analysis of the coefficients a_2 and a_3 is rather difficult; numerical testing leads, however, to the highly probable conclusion that a_3 may never be negative, that is, η_i nay never have a mathematical maximum, which means that it tends mathematically to an infinite positive value at $\mu = 0$. One, therefore, should look for the highest value of η_i, in the whole range of physically possible values of μ, which is not a mathematical maximum.

It should be noticed first that μ cannot drop to zero, but it has a limit

$$\mu_l = \frac{\left(w_1 + w_2\right)c_1''}{w_1c_1'' + w_2c_1'} = \frac{\sigma}{n_1\sigma + 1 - n_1} \tag{21}$$

at which the duct cross-sectional area will become infinitely large, except for initial and final values, if the trivial condition $c_1'' = c_1'$ is not fulfilled in addition. Thus, it must be assumed that

$$\mu \geqq \mu_l$$

and, therefore, the highest value of η_i corresponds to this limit value of μ.

Substituting μ_l as defined by equation (21) into equation (20), the limit value of the efficiency is found:

$$(\eta_i)_l = \frac{2(1 - n_1)^2 \log_e \left(\frac{1}{n_1\sigma + 1 - n_1}\right) + n_1(3n_1 - 2) + 2n_1(1 - n_1)\sigma}{n_1{}^2\left[n_1 + (1 - n_1)\sigma^2\right]} \tag{22}$$

It may be easily proved, for the sake of checking, that, for $0 \leqq n_1 \leqq 1$, $c_1' \geqq 0$, and $c_1'' \geqq 0$, which conditions are physically obvious, the value is always $(\eta_i)_l \leqq 1$. Substituting, namely, $x = \frac{1}{n_1\sigma + 1 - n_1}$, gives the condition

$$\frac{(x - 1)\left[(3 - 2n_1)x - 1\right]}{2(1 - n_1)x^2} \geqq \log_e x$$

To prove that this condition is fulfilled for $0 \leqq x \leqq \frac{1}{1 - n_1}$, that is, for $0 \leqq \sigma \leqq \infty$, it is sufficient to take the first two derivatives of the two sides of the inequality cited above. This inequality becomes an equality only for $x = 1$.

Of course, the extreme case as defined by equations (21) and (22) is not actually applicable; some deviation of numerical values must, therefore, be applied and η_i will become less than $(\eta_i)_l$. In order, however, to have comparative figures on hand, the theoretical case of equation (22) will be studied numerically below, it being compared with the value $(\eta_i)_r$ as applied in the usual routine now (ref. 2),

$$(\eta_i)_r = \frac{\left(w_1c_1' + w_2c_1''\right)^2}{\left(w_1 + w_2\right)\left[w_1\left(c_1'\right)^2 + w_2\left(c_1''\right)^2\right]} = \frac{\left[n_1 + \left(1 - n_1\right)\sigma\right]^2}{n_1 + \left(1 - n_1\right)\sigma^2} \tag{23}$$

It may be easily proved that always

$$(\eta_i)_l \geqq (\eta_i)_r$$

It is sufficient, for instance, to substitute $\frac{1}{n_1\sigma + 1 - n_1} = x$, which leads to the condition

$$\frac{(x - 1)(3x - 1)}{2x^2} \geqq \log_e x$$

Taking the two first derivatives of this inequality, one arrives at the conclusion that $x \geqq 1$, which means that $c_1' \geqq c_1''$. This assumption was made at the very beginning and remains, of course, valid; therefore, the assumption that $(\eta_i)_l \geqq (\eta_i)_r$ is proved, except for $c_1' = c_1''$; that is, $x = 1$ when $(\eta_i)_l = (\eta_i)_r$.

NUMERICAL COMPUTATION OF EQUATIONS (22) AND (23)

The procedure for the numerical calculation of equations (22) and (23) is as follows: Take first the routine formula for efficiency $(\eta_i)_r$. The numerical values of $(\eta_i)_r$ are given in table I and plotted against n_1, for different constant values of σ, in figure 5.

As is seen, $(\eta_i)_r$ passes through a minimum value, the formula for which is easily obtainable by differentiation of equation (23):

$$(\eta_i)_{r_{min}} = \frac{4\sigma}{(1 + \sigma)^2}$$

for

$$(n_1)_e = \frac{\sigma}{1 + \sigma}$$

Numerical values of $(\eta_i)_{r_{min}}$ are given in table II.

Turning now to equation (22), it should be stated that for very small values of n_1 the usual calculation based on form (22) is not sufficiently exact. It is, therefore, better to develop the logarithm as follows:

$$\log_e\left[\frac{1}{1 - (1 - \sigma)n_1}\right] = (1 - \sigma)n_1 + \frac{1}{2}(1 - \sigma)^2 {n_1}^2 + \frac{1}{3}(1 - \sigma)^3 {n_1}^3 + \ldots$$

which gives

$$(\eta_1)_l = \frac{\sigma^2 + \frac{2}{3}(1 - \sigma)(1 + \sigma + \sigma^2)n_1 + \frac{1}{6}(1 - \sigma^2)(1 + 2\sigma + 3\sigma^2){n_1}^2 + \frac{1}{15}(1 - \sigma)^3(1 + 3\sigma + 6\sigma^2){n_1}^3 + \frac{1}{30}(1 - \sigma)^4(1 + 4\sigma + 10\sigma^2){n_1}^4 + \ldots}{\sigma^2 + (1 - \sigma^2)n_1} \tag{24}$$

The numerical values of $(\eta_1)_l$ are given in table III and plotted against n_1, for different constant values of σ, in figure 6.

DUCT FORM

In order to determine the duct form, an additional hypothesis as to changes of c' with the duct length must be made. As has already been pointed out in the section "Study of General Case," a possible assumption is that c' diminishes linearly with the increasing duct length x (see fig. 4). Such an assumption seems to be the simplest and, at the same time, logical enough. Mathematically, it is equivalent to

$$c' = \frac{c_2 - c_1'}{l} x + c_1' \tag{25}$$

In the particular case as defined by equation (18) this leads to:

$$\begin{aligned} A &= \frac{ac' + b}{\gamma_o c'} \\ &= \frac{a(c_2 - c_1')x + (ac_1' + b)l}{\gamma_o\left[(c_2 - c_1')x + c_1' l\right]} \\ &= \frac{\mu\left(w_1c_1'' + w_2c_1'\right)\left[\left(w_1c_1'' + w_2c_1'\right)l - w_2\left(c_1' - c_1''\right)x\right]}{\gamma_o c_1' c_1'' \left\{\mu\left(w_1c_1'' + w_2c_1'\right)l - \left[\mu\left(w_1c_1'' + w_2c_1'\right) - \left(w_1 + w_2\right)c_1''\right]x\right\}} \end{aligned} \tag{26}$$

where μ must, of course, be chosen a little larger than the limit value defined by equation (21), but close to it, in order to maintain the highest possible efficiency. The length l will be evaluated according to practical considerations, as no theory may be advanced here in this connection. The aim is, of course, the thorough mixing with minimum losses and the obtaining of complete uniformization of velocity at (or before) the final cross section.

As for the cross-sectional area A, it may be assumed in any form whatsoever. If a circular cross section is assumed, the diameter will be $D = \sqrt{\frac{4}{\pi} A}$.

In the limit case $\mu = \mu_l$, the results become, of course, trivial; that is,

$$A = A_1 = A_2 = \text{Constant} = \frac{(w_1 + w_2)}{c_1'}$$

$$c_2 = c_1'' = c_1'$$

The portion l of the duct lying between cross sections A_1 and A_2 cannot be regarded, in general, as a whole unit, either in the case of an injector or in other cases. The reason is that the "final" pressure p_2 is, in general, not equal to p_1. Neither does it equal the pressure p_o of the surroundings. In the case of an injector used for pumping purposes, it may be assumed that $p_1 = p_o$, while the final pressure should be raised as high as possible, for example, to p_3, by applying the feasible minimum c_3 of the final velocity. This may be done only by adding a diffuser extending from A_2 to A_3.

The other possible case is a thrust producer. Here not only $p_1 = p_o$ but also $p_3 = p_o$ should be assumed.

In the general case, according to equation (16),

$$J = \eta_i(c_1')^2\left[n_1 + (1 - n_1)\sigma^2\right]$$

and, according to equations (15a) and (17):

$$p_2 - p_1 = p_2 - p_o = \frac{\gamma_o}{2g}\left(J - c_2^2\right)$$

wherefrom

$$p_2 - p_1 = p_2 - p_o = \frac{\gamma_o}{2g}\left\{\eta_i\left[n_1 + (1 - n_1)\sigma^2\right](c_1')^2 - c_2^2\right\} \tag{27}$$

In the additional diffuser (see fig. 7) the pressure is brought back to p_o, that is, $p_3 = p_o$. Thus

$$p_3 - p_2 = -\left(p_2 - p_o\right) = \frac{\gamma_o}{2g}\left\{c_2{}^2 - \eta_i\left[n_1 + \left(1 - n_1\right)\sigma^2\right]\left(c_1{}'\right)^2\right\}$$

But, according to Bernoulli's law, if friction and other losses are disregarded,

$$\frac{p_3 - p_2}{\gamma_o} + \frac{c_3{}^2}{2g} - \frac{c_2{}^2}{2g} = 0$$

Thus:

$$c_3 = \sqrt{c_2{}^2 - \frac{2g}{\gamma_o}\left(p_3 - p_2\right)}$$

$$= c_1{}'\sqrt{\eta_i\left[n_1 + \left(1 - n_1\right)\sigma^2\right]} \tag{28}$$

In the particular case as defined by equation (18), the value

$$c_2 = \frac{\sigma c_1{}'}{\mu\left[1 - \left(1 - \sigma\right)n_1\right]}$$

should be entered into equation (27), and the value of η_i as defined by equations (19) and (20) should be entered into equation (28).

The area of the final cross section of the diffuser is, obviously,

$$A_3 = \frac{w_1 + w_2}{\gamma_o c_3} = \frac{w_1 + w_2}{\gamma_o c_1{}'\sqrt{\eta_i\left[n_1 + \left(1 - n_1\right)\sigma^2\right]}} \tag{29}$$

The particular case of an ejector thrust augmenter with constant-pressure mixing zone is studied more thoroughly in the appendix.

Ecole Polytechnique (University of Montreal),
Montreal, Quebec, Canada, January 13, 1949.

APPENDIX

THEORY OF "CONSTANT-PRESSURE" EJECTOR THRUST AUGMENTER

In the computation developed in the main text the case of an injector has been treated almost exclusively, with the aim of obtaining the dynamic compression. The ultimate goal has been to get the largest possible total pressure (i.e., a sum of static pressure and velocity head). The case of an ejector thrust augmenter is different in that the ultimate goal is to get the largest possible thrust, depending directly on momentum, under given initial and final conditions. This case is briefly investigated theoretically below.

Although the definition of efficiency is a little different in this case, it remains intimately related to the final momentum of the mixture leaving the ejector; therefore, all the general remarks concerning the ejector (mixing zone) throughout the paper remain valid. However, as the investigation of the most general case would be rather involved, it seems more practical to investigate first the particular case of the ejector, that is, of the mixing zone, at constant pressure. Therefore formulas (8) and (9), applying to the mixing zone, will remain valid.

Obviously, if the given initial conditions p_o,c_o' for the primary stream and p_o,c_o'' for the secondary stream are applied directly at the ejector exit, that is, if $p_1 = p_o' = p_o''$, $c_1' = c_o'$, and $c_1'' = c_o''$, and if, in addition, $p_o = p_1 = p_2$ (or atmospheric pressure, thus no diffuser or nozzle at station 2), there is no thrust augmentation whatsoever, because the momentum remains unchanged. It therefore follows that the given initial conditions should be first subject to appropriate changes if any thrust augmentation is to be obtained with a constant-pressure ejector. These changes may be either a prior acceleration (by nozzles) or deceleration (by diffusers) before the two currents enter the ejector, with the condition $p_1' = p_1'' = p_1$. Of course, in such a case a diffuser (or nozzle) must be applied downstream of the ejector, in order to establish the desired final pressure $p_3 = p_o$.

For the given initial conditions p_o',c_o' and p_o'',c_o'' and the final pressure p_o, there exists a certain value of $p_1 = p_2$ yielding a maximum of thrust, which may exceed that obtained with the primary stream alone and without ejector. Of course, these optimal values of p_1 and Th depend on the mass-flow ratio, that is, on $n_1 = \dfrac{w_1}{w_1 + w_2}$.

In order to define the problem further, assume the system as shown in figure 8, in which the boundary losses in the ejector and the exit diffuser are disregarded.

The given values are:

p_o atmospheric pressure, lb/sq ft

c_o velocity of travel, ft/sec

n_1 mass-flow ratio, $\dfrac{w_1}{w_1 + w_2}$

γ_o specific gravity, lb/cu ft

H actual head produced by pump, ft

The value of c_1' is to be found for $(Th)_{max}$; thus values of p_1 and c_1'' will thereby be defined.

The following formulas follow:

$$H = -\frac{1}{\gamma_o}(p_o - p_1) + \frac{1}{2g}\left[(c_1')^2 - (c_o'')^2\right] = \frac{1}{2g}\left[(c_o')^2 - (c_o'')^2\right]$$

$$c_1'' = \sqrt{c_o^2 + \frac{2g}{\gamma_o}(p_o - p_1)}$$

$$c_2 = \frac{w_1 c_1' + w_2 c_1''}{w_1 + w_2} = n_1 c_1' + (1 - n_1)c_1''$$

$$c_3 = \sqrt{c_2^2 - \frac{2g}{\gamma_o}(p_o - p_1)}$$

$$Th = \frac{1}{g}(w_1 + w_2)(c_3 - c_o)$$

The (external) propulsive efficiency is

$$\eta_{ex} = \frac{c_o(Th)}{w_1 H}$$

If the ejector were not applied, the primary stream alone would produce the thrust

$$(Th)_o = \frac{1}{g} w_1\left(\sqrt{2gH + c_o{}^2} - c_o\right) = \frac{1}{g} w_1\left[\sqrt{(c_1')^2 - \frac{2g}{\gamma_o}(p_o - p_1)} - c_o\right]$$

Introducing the dimensionless factors

$$p_1/p_o = \pi$$

$$\frac{\sqrt{2gH}}{\sqrt{2g\frac{p_o}{\gamma_o}}} = \lambda$$

$$\frac{c_1'}{\sqrt{\frac{2g}{\gamma_o} p_o}} = \sigma_1'$$

$$\frac{c_o}{\sqrt{\frac{2g}{\gamma_o} p_o}} = \sigma_o$$

$$\frac{Th}{(Th)_o} = \zeta$$

the following expressions are obtained:

$$\pi + \left(\sigma_1'\right)^2 - \sigma_o{}^2 - 1 = \lambda^2$$

$$c_1'' = \sqrt{\frac{2g}{\gamma_o} p_o}\left[n_1\sigma_1' + (1 - n_1)\sqrt{(\sigma_1')^2 - \lambda^2}\right]$$

$$c_3 = \sqrt{\frac{2g}{\gamma_o} p_o}\sqrt{\lambda^2 - (\sigma_1')^2 + \sigma_o{}^2 + \left[n_1\sigma_1' + (1 - n_1)\sqrt{(\sigma_1')^2 - \lambda^2}\right]^2}$$

$$\mathrm{Th} = \left(w_1 + w_2\right)\sqrt{\frac{2p_o}{g\gamma_o}}\left\{\sqrt{\lambda^2 - \left(\sigma_1'\right)^2 + \sigma_o^2 + \left[n_1\sigma_1' + \left(1 - n_1\right)\sqrt{\left(\sigma_1'\right)^2 - \lambda^2}\right]^2} - \sigma_o\right\}$$

$$(\mathrm{Th})_o = w_1\sqrt{\frac{2p_o}{g\gamma_o}}\left(\sqrt{\lambda^2 + \sigma_o^2} - \sigma_o\right)$$

$$\zeta = \frac{\sqrt{\lambda^2 - \left(\sigma_1'\right)^2 + \sigma_o^2 + \left[n_1\sigma_1' + \left(1 - n_1\right)\sqrt{\left(\sigma_1'\right)^2 - \lambda^2}\right]^2} - \sigma_o}{n_1\left(\sqrt{\lambda^2 + \sigma_o^2} - \sigma_o\right)}$$

$$\eta_{ex} = \frac{2\sigma_o}{n_1\lambda^2}\left\{\sqrt{\lambda^2 - \left(\sigma_1'\right)^2 + \sigma_o^2 + \left[n_1\sigma_1' + \left(1 - n_1\right)\sqrt{\left(\sigma_1'\right)^2 - \lambda^2}\right]^2} - \sigma_o\right\}$$

$$= \frac{2\sigma_o\left(\sqrt{\lambda^2 + \sigma_o^2} - \sigma_o\right)}{\lambda^2}\xi$$

As is seen, ζ is a function of four different factors, n_1, σ_o, λ, and σ_1'; logically, they are limited as follows:

$$0 \leqq n_1 \leqq 1$$

$$0 \leqq \sigma_o \leqq \infty$$

$$0 \leqq \lambda \leqq \infty$$

$$0 \leqq \pi \leqq \infty$$

where it is understood that

$$\lambda \leqq \sigma_1' \leqq \sqrt{\lambda^2 + \sigma_o^2 + 1}$$

$$0 \leqq \pi \leqq \lambda^2 + \sigma_o^2 + 1$$

and therefore

$$\lambda^2 - \left(\sigma_1'\right)^2 + \sigma_o^2 + 1 \geqq 0$$

In order to determine the influence of these four factors, the formula for ζ should be examined first with respect to each of them separately. The condition $d\zeta/dn_1 = 0$ leads to

$$\left(n_1\right)_e = 0$$

It may be shown that for $n_1 = 0$

$$\zeta_{n_1=0} = \frac{\left(\sigma_1' - \sqrt{\left(\sigma_1'\right)^2 - \lambda^2}\right)\sqrt{\left(\sigma_1'\right)^2 - \lambda^2}}{\sigma_o\left(\sqrt{\lambda^2 + \sigma_o^2} - \sigma_o\right)} > 1 \quad \text{if} \quad \pi \leqq 1$$

except for $p_1 = p_o$ $\left(\text{i.e.,}\ \lambda^2 = \left(\sigma_1'\right)^2 - \sigma_o^2\right)$, or for $\lambda = 0$ where $\zeta_{n_1=0} = 1$. This is a maximum because, for $n_1 = 1$, $\zeta = 1$. Therefore, the secondary mass flow should be as large as possible.

The condition $d\zeta/d\sigma_1' = 0$ does not lead to any extremal values, simply yielding $\lambda = 0$, which does not impose any condition upon σ_1'. It may be shown that, in general, for $\lambda = 0$,

$$\zeta_{\lambda=0} = 1$$

$$\left(\eta_{ex}\right)_{\lambda=0} = 1$$

It may also be shown that, for $p_1 = p_o$,

$$\zeta_{\pi=1} = 1$$

$$\left(\eta_{ex}\right)_{\pi=1} = \frac{2\sigma_o}{\sigma_1' + \sigma_o}$$

as well as that, for $n_1 = 0$,

$$\left(\eta_{ex}\right)_{n_1=0} = \frac{2}{\lambda^2}\left[\sigma_1' - \sqrt{\left(\sigma_1'\right)^2 - \lambda^2}\right]\sqrt{\left(\sigma_1'\right)^2 - \lambda^2} \leqq 1 \quad \text{if} \quad \lambda \gtrapprox 0$$

Furthermore, ζ may become very high for small values of σ_o (the efficiency being low, however), whereas it attains certain maximum value (greater than 1) at a definite value of λ. The picture is as shown in figure 9.

Finally, it may be shown, in general, that $\zeta > 1$ for $0 < \pi < 1$ and $0 < n_1 < 1$.

If the ejector is not applied, then $\pi = 1$, $n_1 = 1$, $\zeta_o = 1$, and $\left(\eta_{ex}\right)_o = \dfrac{2\sigma_o}{\sigma_1' + \sigma_o}$. Therefore, the ratio of the increase in efficiency by an ejector is

$$\begin{aligned}
\epsilon &= \frac{\eta_{ex}}{\left(\eta_{ex}\right)_o} \\
&= \frac{\sigma_1' + \sigma_o}{\lambda^2}\left(\sqrt{\lambda^2 + \sigma_o^2} - \sigma_o\right)\zeta \\
&= \frac{\left(\sigma_1' + \sigma_o\right)}{n_1\lambda^2}\left\{\sqrt{\lambda^2 - \left(\sigma_1'\right)^2 + \sigma_o^2 + \left[n_1\sigma_1' + \left(1 - n_1\right)\sqrt{\left(\sigma_1'\right)^2 - \lambda^2}\right]^2} - \sigma_o\right\}
\end{aligned}$$

If the values of σ_{1_o}' and λ_o, applied without the ejector, are going to be changed (e.g., λ will increase) while σ_o remains the same, then

$$\begin{aligned}
\epsilon' &= \frac{\sigma_{1_o}' + \sigma_o}{n_1\lambda^2}\left\{\sqrt{\lambda^2 - \left(\sigma_1'\right)^2 + \sigma_o^2 + \left[n_1\sigma_1' + \left(1 - n_1\right)\sqrt{\left(\sigma_1'\right)^2 - \lambda^2}\right]^2} - \sigma_o\right\} \\
&= \frac{\sigma_{1_o}' + \sigma_o}{n_1\lambda^2}\left\{\sqrt{-1 + \pi + \left[n_1\sqrt{\lambda^2 + \sigma_o^2 + 1 - \pi} + \left(1 - n_1\right)\sqrt{\sigma_o^2 + 1 - \pi}\right]^2} - \sigma_o\right\}
\end{aligned}$$

In the applications of a centrifugal pump to the dynamic propulsion, the requirement of a reasonable efficiency imposes a rather large mass flow while the head should be kept small. This results in huge tubing diameters. In order to improve this situation, an ejector thrust augmenter may be applied in such a manner as to keep the efficiency unchanged, while the pump head λ is increased with subsequent reduction in mass flow, if the power supply remains unchanged. The problem may thus be defined as follows: For the given $\epsilon' = 1$, increase λ and change π so as to maintain a sufficiently high value of thrust, while the velocity of travel σ_o remains unchanged.

The condition $\epsilon' = 1$ leads to

$$(1 - \pi) = \frac{1}{4(1 - n_1)\left(\sigma_{1_o}' + \sigma_o\right)^2\left[\left(\sigma_{1_o}'\right)^2 - \sigma_o^2 - n_1\lambda^2\right]}\left\{n_1^2\lambda^6 + 2n_1\left(\sigma_{1_o}' + \sigma_o\right)\left[\left(2 - n_1\right)\sigma_o - n_1\sigma_{1_o}'\right]\lambda^4 + \right.$$

$$\left.\left(\sigma_{1_o}' + \sigma_o\right)^2\left[\left(4 - 3n_1^2\right)\sigma_o^2 - 2n_1(2 - n_1)\sigma_{1_o}'\sigma_o + n_1^2\left(\sigma_{1_o}'\right)^2\right]\lambda^2 - 4\left(1 - n_1\right)\left(\sigma_{1_o}' + \sigma_o\right)^2\left[\left(\sigma_{1_o}'\right)^2 - \sigma_o^2\right]\sigma_o^2\right\}$$

wherefrom the functional dependence of π on λ may be found for assumed numerical values of n_1, σ_{1_o}', and σ_o. The thrust will remain constant because the power supply remains the same; that is, $w_{1_o}2gH_o = w_1 2gH$. Therefore,

$$\frac{w_1\zeta}{w_{1_o}\zeta_o} = \frac{w_1\lambda^2}{w_{1_o}\left(\sigma_{1_o}' + \sigma_o\right)\left(\sqrt{\lambda_o^2 + \sigma_o^2} - \sigma_o\right)}$$

$$= \frac{w_1\lambda^2}{w_{1_o}\left(\sigma_{1_o}^2 - \sigma_o^2\right)}$$

$$= \frac{w_1\lambda^2}{w_{1_o}\lambda_o^2}$$

$$= 1$$

Let $p_o = 2{,}120$ pounds per square foot, $\gamma_o = 62.5$ pounds per cubic foot, $c_o = 15$ feet per second, $H_o = 15$ feet, $\pi_o = 1$ and $n_1 = 0.25$.

Then $\sigma_o = 0.320$, $\lambda_o = 0.662$, $\sigma_{1_o}' = 0.735$, $c_{1_o}' = 34.54$ feet per second, $w_2 = 3w_1$, and

$$1 - \pi = \frac{0.01872\lambda^6 + 0.05944\lambda^4 + 0.0767\lambda^2 - 0.04483}{0.438 - 0.25\lambda^2}$$

$$\frac{\sigma_1'}{\sigma_{1_o}'} = \frac{\sqrt{\lambda^2 + (1 - \pi) + 0.1024}}{0.735}$$

The reduction of the exhaust tube will be in proportion

$$\delta = \sqrt{\frac{w_1 \sigma_{1_o}'}{w_{1_o} \sigma_1'}}$$

where

$$\frac{w_1}{w_{1_o}} = \frac{\lambda_o^2}{\lambda^2}$$

Some calculated values of $1 - \pi$ and δ for given values of λ^2 are presented in the following table:

λ^2	0.438	0.5	0.6	0.7	0.8	0.9	1.0	1.1
$1 - \pi$	0	0.0342	0.090	0.169	0.269	0.404	0.585	0.836
w_1/w_{1_o}	1	0.876	0.730	0.626	0.5475	0.4865	0.438	0.398
σ_1'/σ_{1_o}'	1	1.086	1.210	1.340	1.472	1.613	1.766	1.941
δ	1	0.898	0.776	0.684	0.610	0.549	0.498	0.4527

For the same assumptions but with $n_1 = 0.2$,

$$1 - \pi = \frac{0.01124\lambda^6 + 0.0508\lambda^4 + 0.07794\lambda^2 - 0.04483}{0.438 - 0.2\lambda^2}$$

for the following values of λ^2:

λ^2	0.438	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3
$1 - \pi$	0	0.0244	0.0712	0.1289	0.2002	0.2896	0.400	0.539	0.713	0.939
w_1/w_{1_o}	1	0.876	0.730	0.626	0.5475	0.4865	0.438	0.398	0.3648	0.3368
σ_1'/σ_{1_o}'	1	1.078	1.196	1.312	1.429	1.547	1.668	1.795	1.930	2.080
δ	1	0.902	0.782	0.690	0.619	0.559	0.5125	0.471	0.435	0.4025

Finally, for $n_1 = 0.4$,

$$1 - \pi = \frac{0.0599\lambda^6 + 0.0689\lambda^4 + 0.061\lambda^2 - 0.04483}{0.438 - 0.4\lambda^2}$$

for the following values of λ^2:

λ^2	0.438	0.5	0.6	0.7	0.8	0.85
$1 - \pi$	0	0.0436	0.1489	0.3302	0.667	0.955
w_1/w_{1_o}	1	0.876	0.730	0.626	0.5475	0.5150
σ_1'/σ_{1_o}'	1	1.093	1.256	1.447	1.704	1.877
δ	1	0.895	0.7625	0.658	0.566	0.524

The above numerical results are presented graphically in figures 10(a) and 10(b).

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2. Gosline, James E., and O'Brien, Morrough P.: The Water Jet Pump. Univ. of Calif. Pub. in Eng., vol. 3, no. 3, 1933, pp. 167-190.

TABLE I

VALUES OF $\left(\eta_i\right)_r$

σ \ η_1	$\left(\eta_i\right)_r$										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.000
0.1	1.000	0.331	0.3768	0.446	0.5215	0.599	0.678	0.758	0.838	0.919	1.000
0.2	1.000	0.576	0.5585	0.590	0.6375	0.6925	0.750	0.811	0.874	0.936	1.000
0.3	1.000	0.756	0.712	0.716	0.741	0.775	0.815	0.858	0.904	0.952	1.000
0.4	1.000	0.8675	0.825	0.8165	0.826	0.8445	0.870	0.899	0.931	0.965	1.000
0.5	1.000	0.931	0.900	0.890	0.891	0.900	0.915	0.9325	0.953	0.975	1.000
0.6	1.000	0.966	0.9475	0.939	0.938	0.9415	0.949	0.958	0.970	0.985	1.000
0.7	1.000	0.985	0.975	0.970	0.969	0.970	0.973	0.977	0.984	0.9915	1.000
0.8	1.000	0.995	0.9915	0.9885	0.988	0.988	0.9885	0.990	0.993	0.996	1.000
0.9	1.000	0.9985	0.998	0.9975	0.9972	0.997	0.9975	0.998	0.9985	0.999	1.000
1.0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

TABLE II

MINIMUM VALUES OF $(\eta_i)_r$

σ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.00
$(\eta_1)_e$	0	0.0909	0.1667	0.2308	0.286	0.3333	0.375	0.412	0.445	0.474	0.500
$(\eta_i)_{r_{min}}$	0	0.3306	0.556	0.710	0.816	0.889	0.9375	0.969	0.9875	0.997	1.000

TABLE III

VALUES OF $\left(\eta_i\right)_l$

σ \ η_1	$\left(\eta_i\right)_l$										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.00
0	0.666(6)	0.6840	0.7030	0.7235	0.7460	0.7720	0.8015	0.8355	0.8770	0.9270	1.0000
0.1	1.0000	0.7190	0.7230	0.7380	0.7575	0.7825	0.8125	0.8460	0.8840	0.9220	1.0000
0.2	1.0000	0.7930	0.7726	0.7768	0.7890	0.8092	0.8350	0.8650	0.9008	0.9444	1.0000
0.3	1.0000	0.8686	0.8375	0.8250	0.8315	0.8470	0.8655	0.8900	0.9200	0.9565	1.0000
0.4	1.0000	0.9170	0.8880	0.8760	0.8760	0.8840	0.8970	0.9164	0.9400	0.9685	1.0000
0.5	1.0000	0.9520	0.9285	0.9180	0.9170	0.9214	0.9290	0.9412	0.9560	0.9765	1.0000
0.6	1.0000	0.9740	0.9600	0.9520	0.9495	0.9500	0.9550	0.9622	0.9720	0.9850	1.0000
0.7	1.0000	0.9770	0.9790	0.9750	0.9725	0.9726	0.9750	0.9790	0.9840	0.9915	1.0000
0.8	1.0000	0.9950	0.9910	0.9894	0.9880	0.9880	0.9890	0.9910	0.9935	0.9965	1.0000
0.9	1.0000	0.9990	0.9975	0.9960	0.9955	0.9960	0.9965	0.9980	0.9985	0.9990	1.0000
1.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

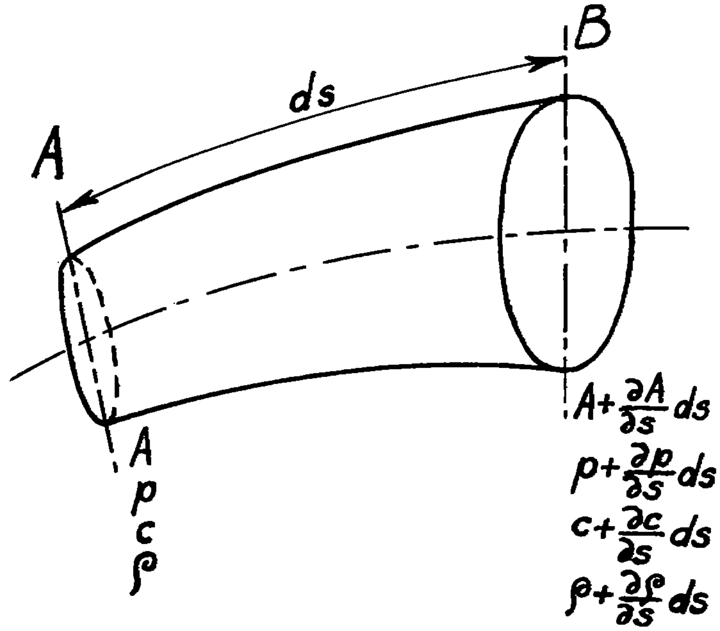


Figure 1.- Infinitesimal element of a flow vein.

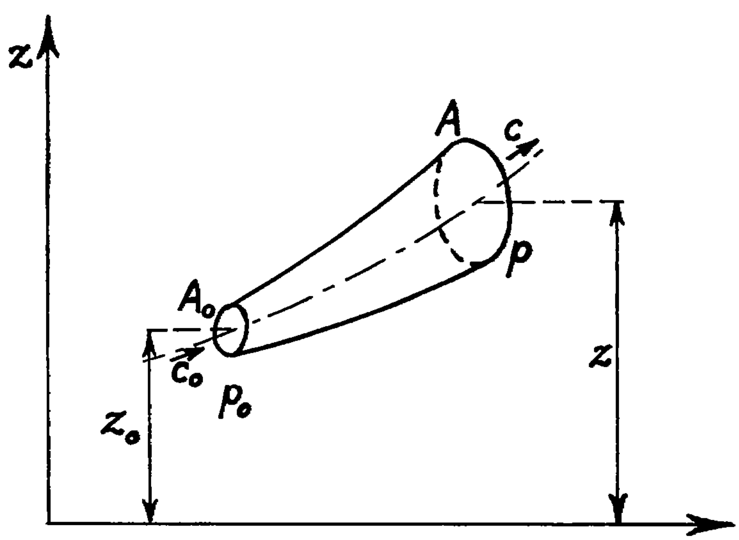


Figure 2.- A flow vein in gravitational field.

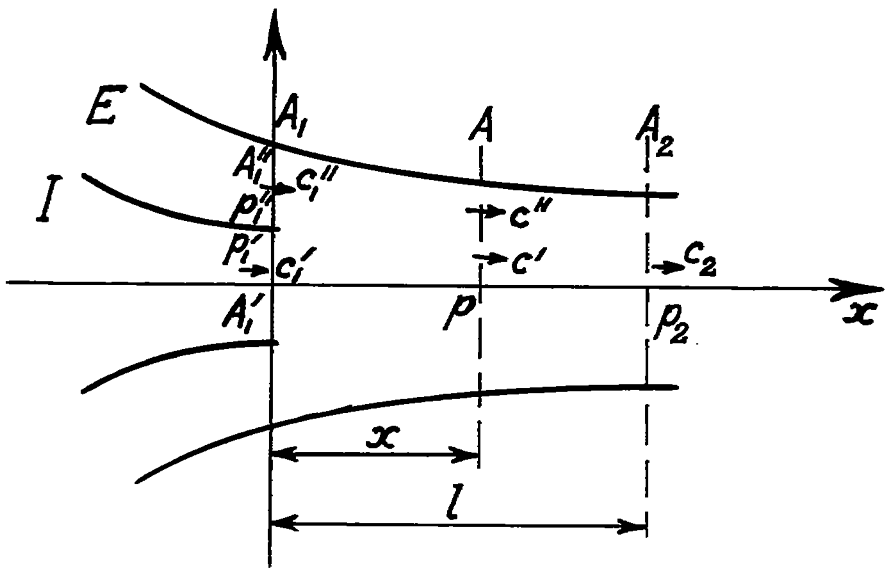


Figure 3.- Illustrative diagram of jet syphon.

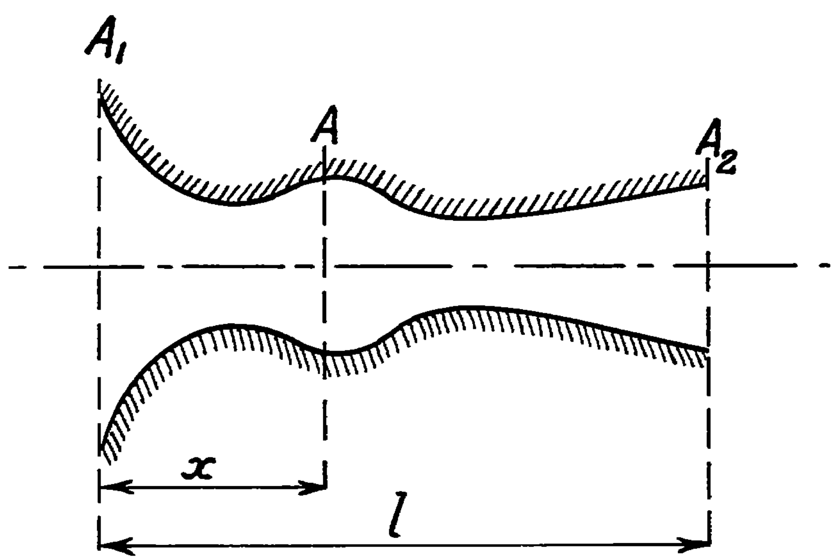


Figure 4.- Arbitrarily shaped mixing zone of jet syphon.

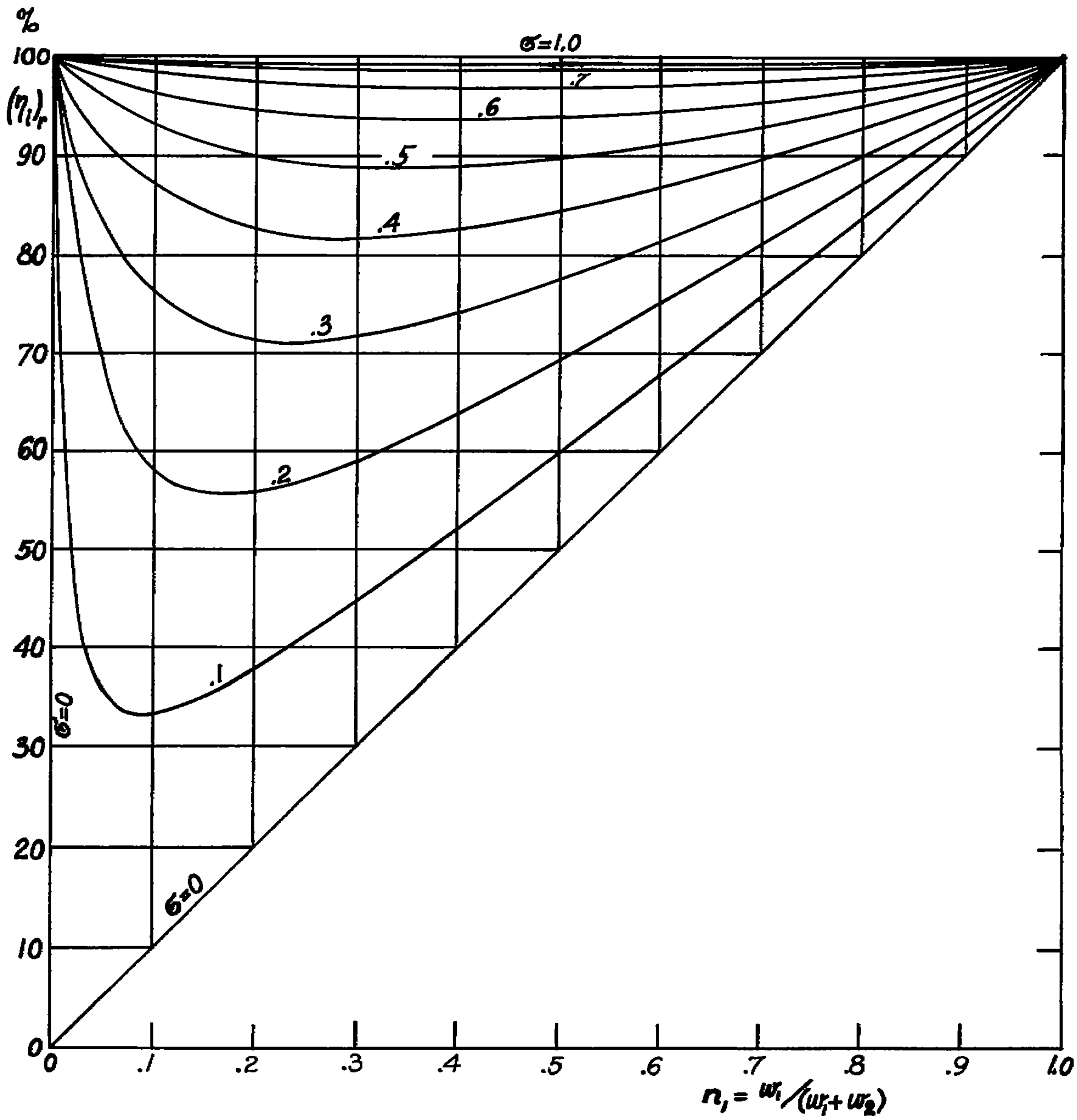


Figure 5.- Graphical representation of injector efficiency as computed with conventional formula.

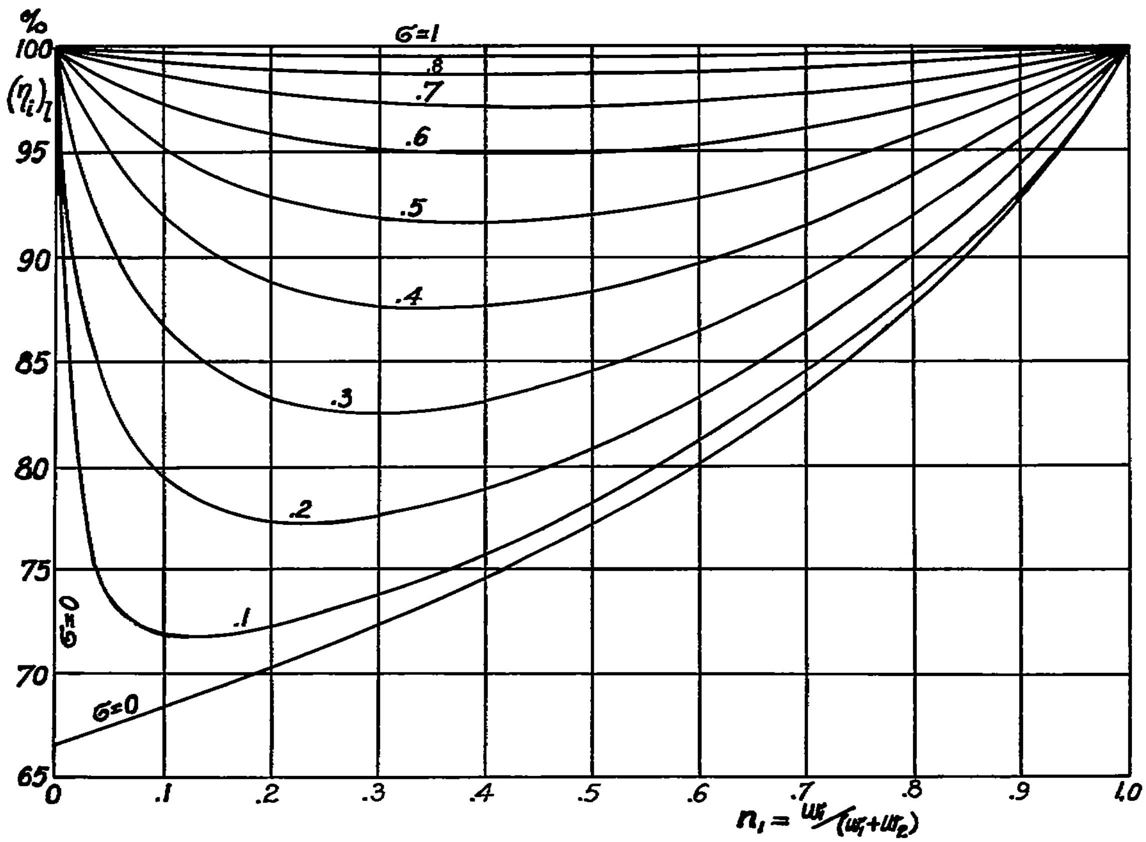


Figure 6.- A case of injector efficiency as computed according to new theory.

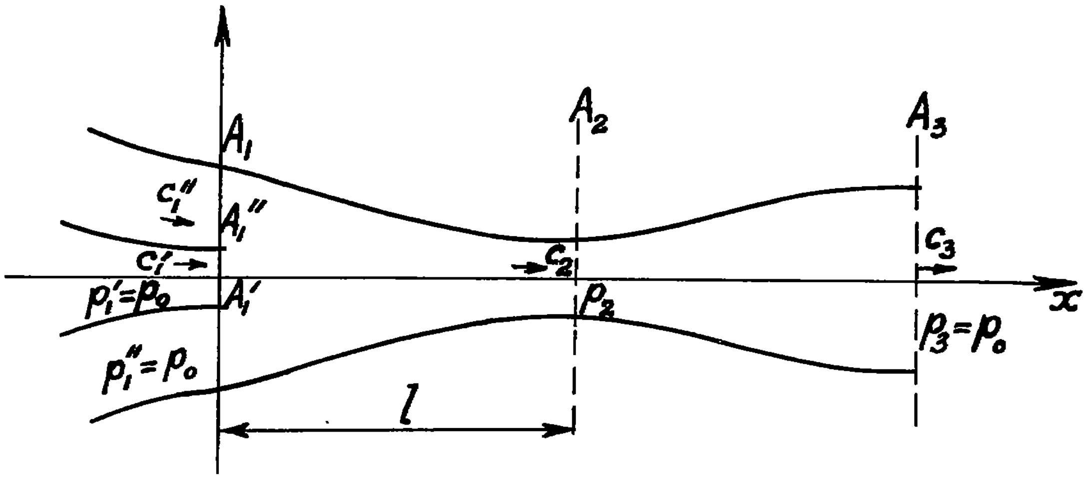


Figure 7.- Illustrative diagram of ejector thrust augmenter.

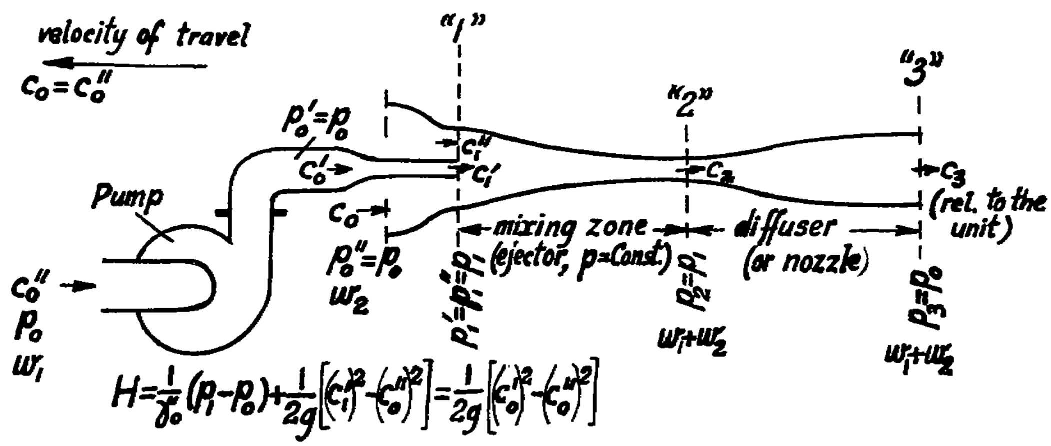


Figure 8.- Illustrative diagram of constant-pressure ejector thrust augmenter.

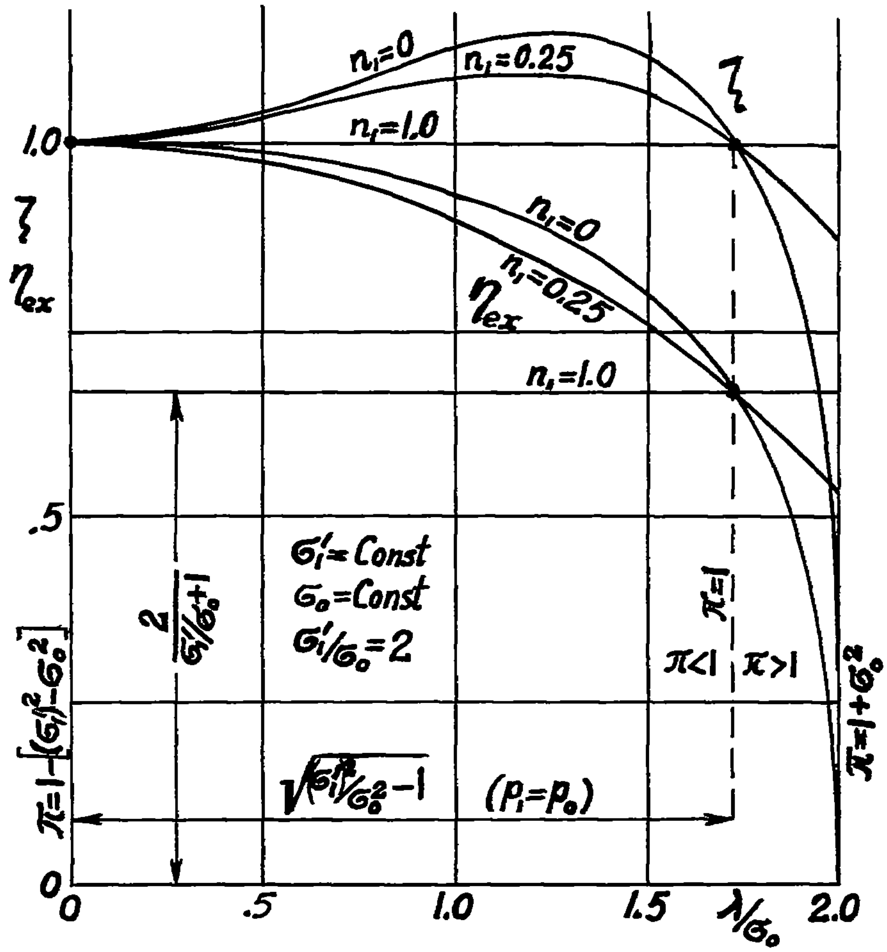
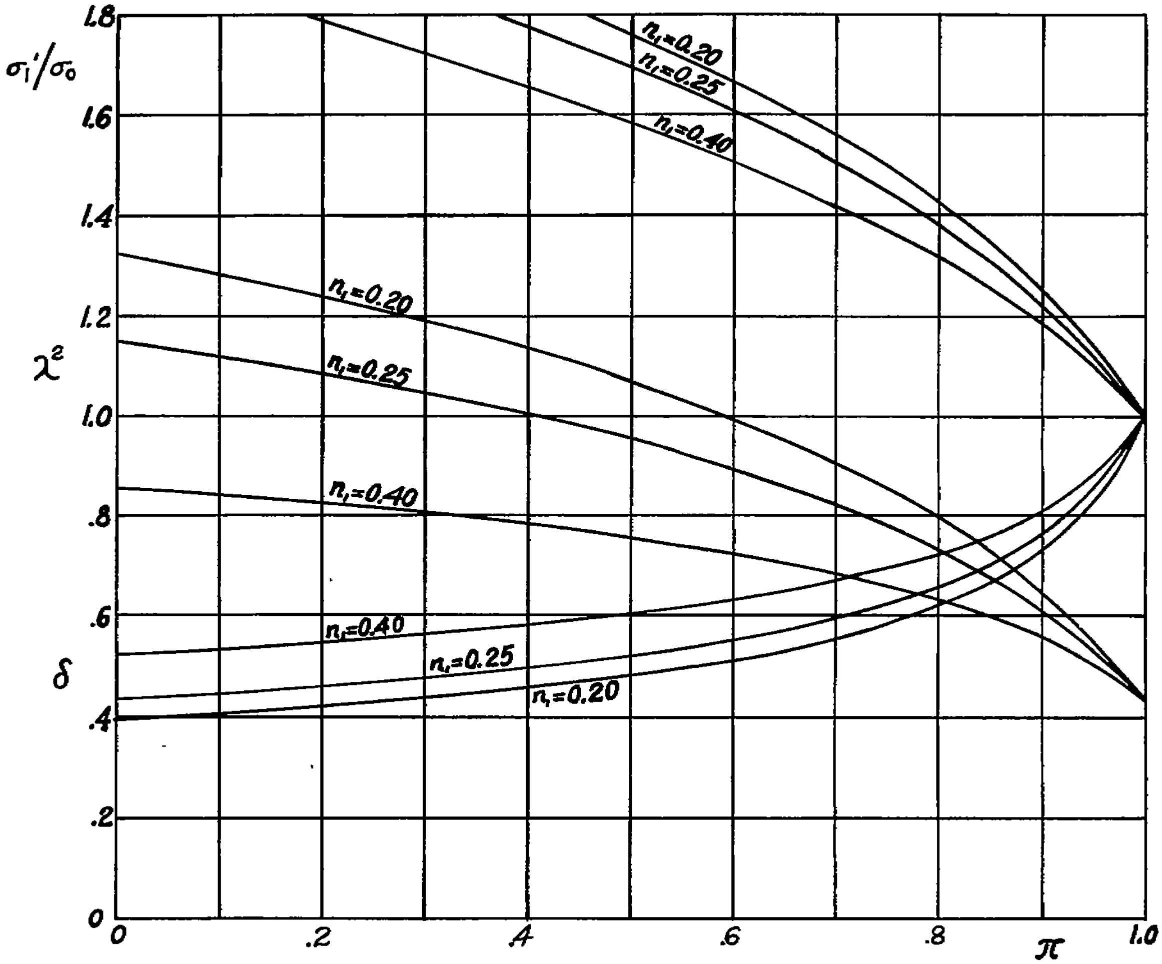
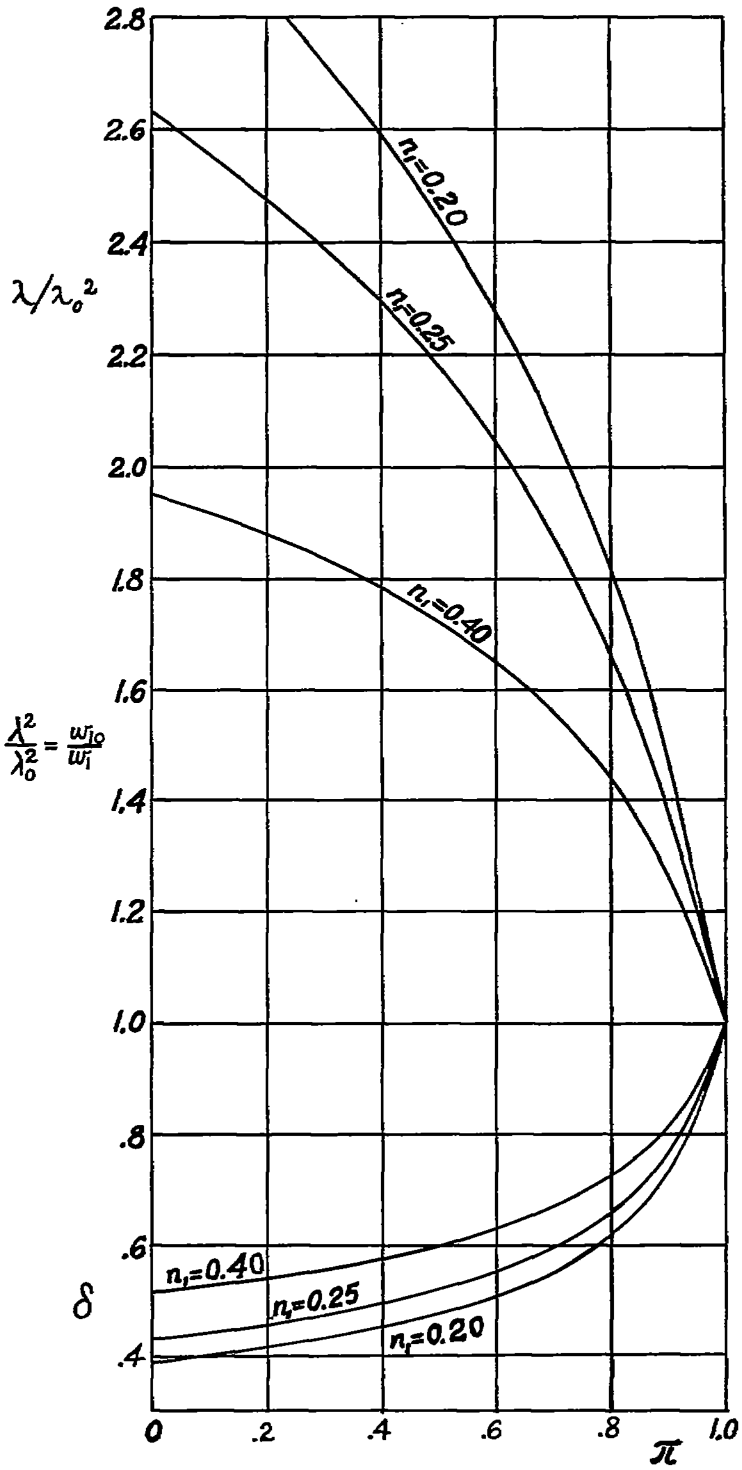


Figure 9.- Graph for efficiency and thrust of constant-pressure ejector thrust augmenter.



(a) Variation of σ_1'/σ_0, δ, and λ^2 with π.

Figure 10.- Dimensionless characteristics of constant-pressure ejector thrust augmenter.



(b) Variation of λ^2/λ_o^2 and δ with π.

Figure 10.- Concluded.